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Stabilizability of a class of stochastic bilinear hybrid systems ☆

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ABSTRACT

The problem of the stabilizability of stochastic nonlinear hybrid systems with a Markovian or any switching rule is considered. Using the Lyapunov technique sufficient conditions for the asymptotic stabilizability in probability by a smooth controller in every structure are found. In particular, the asymptotic stabilizability in probability problem of stochastic bilinear hybrid systems with a Markovian or any switching rule is discussed and a closed-loop controller is found. Also the sufficient conditions for the exponential mean-square stabilizability for bilinear hybrid systems with any switching based on the Lie algebra approach are formulated and an open-loop controller is designed. The obtained results are illustrated by examples and simulations.

1. Introduction

The problem of the stabilizability of dynamic systems is one of the basic problems in the control theory. To describe the uncertain parameters and excitations appearing in real dynamic systems in corresponding models usually the stochastic differential equations are used. Developments on the stabilizability of stochastic dynamic systems can be found, for instance in monographs [16,11,12,25,29]. An important class of nonlinear control systems, called bilinear control systems, are systems described by stochastic differential equations containing terms of products of state and control variables. The study of bilinear systems began in the late 1960s and has continued from its need in applications (many real world systems appearing in economy, biology, chemistry, biochemistry, physics and engineering can be approximated by bilinear models), see for instance, [20,7,21,10]. The stabilizability of a special class of nonlinear systems described by stochastic differential equations containing terms of products of nonlinear functions of state variables and control variables has been studied in past years in [13,14,2,18,26]. In papers [13,14,2] a stabilizing smooth state feedback law has been designed explicitly while in [18] an open-loop controller was found.

One of the most important class of control systems are switching and hybrid systems which are dynamic systems consisting of a number of structures described by deterministic or stochastic differential equations. In the successive moments of time their structures can change according as well to the given switching rule as randomly thereupon creates a hybrid system. The qualitative and quantitative analysis of hybrid systems is well established and was summarized for instance, in books [5,17,3,8].

The problem of the stability and the stabilizability of deterministic and stochastic hybrid systems was analyzed by many authors. To obtain the sufficient conditions of the stability of these systems, mainly the Lyapunov approach was used, for instance in the case of stochastic hybrid systems it was done in [4,30,19,9,15]. In the case of deterministic hybrid systems also another methods based on the Lie algebra approach [6] were considered by Agrachev and Liberzon [1], Liberzon [17],

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Zhai et al. [27,28]. The first application of the Lie algebra approach to the study of the exponential mean-square stability of stochastic linear hybrid systems with any switching rule was proposed by authors in [23,24].

The objective of this paper is to extend results of Florchinger [14] to stochastic control hybrid systems with two types of the switching law: a Markovian or any switching rule, by using Lyapunov machinery introduced by Khasminskii, Zhu and Yin [15] (the Markovian switching rule case) or the common Lyapunov approach [17] (any switching rule case) together with results of Florchinger for nonhybrid systems [14]. We have found the sufficient conditions for the asymptotic stabilizability in probability and have designed a stabilizing smooth state feedback law in every structure of the hybrid system explicitly. We have discussed the bilinear control systems more precisely and we have obtained sufficient conditions for the asymptotic stabilizability in probability. We have also found an explicit form of a closed-loop controller.

The next aim of this paper is to use results obtained in [24] to find the sufficient conditions of the exponential mean-square stabilizability of stochastic bilinear control hybrid systems with any switching and to find an open-loop controller. The Lie algebra approach is one of the most important mathematical tool in the stabilizability analysis in this case [17,27, 28]. Note that the similar model for nonhybrid stochastic systems has been considered by Luesink and Nijmeijer in [18].

The paper is organized as follows. In Section 2, we introduce the notation and recall some basic facts, theorems and definitions concerning the exponential mean-square stability and the asymptotic stability in probability of a null solution of a stochastic differential equation. In Section 3 we derive sufficient conditions for the stabilizability of a class of nonlinear stochastic hybrid systems with two types of switching rule: a Markovian or any switching rule and we obtain an explicit form of controllers. In Section 4 we deal with bilinear hybrid systems (with any switching or with a Markovian switching rule). We consider two types of controllers: open- and closed-loop. We derive the sufficient conditions for the exponential mean-square stabilizability of control hybrid systems with any switching and we obtain an explicit form of an open-loop controller. We also give sufficient conditions for the asymptotic stabilizability in probability of control bilinear hybrid systems with a Markovian or any switching rule and we find closed-loop controllers for the problems. Illustrative examples and simulations are also shown.

2. Mathematical preliminaries

Throughout this paper we use the following notation. Let $|\cdot|$ and $\langle\cdot\rangle$ be the Euclidean norm and the inner product in $\mathbb{R}^n$, respectively. By $\lambda(\mathbf{A})$ we denote the eigenvalue of the matrix $\mathbf{A}$, $\lambda_{\min}(\mathbf{A})$ and $\lambda_{\max}(\mathbf{A})$ denotes the smallest and the biggest eigenvalue of the matrix $\mathbf{A}$, respectively; $\operatorname{tr}(\mathbf{A})$ denotes a trace of the matrix $\mathbf{A}$. We denote the indicator function of a set G by $\mathbb{I}_G$. We mark $\mathbb{R}_+ = [0,\infty)$, $\mathbb{T} = [t_0,\infty)$, $t_0 \geqslant 0$. $\mathbb{S} = \{1,\dots,N\}$ is the set of states. The process $\sigma(t): \mathbb{R}_+ \to \mathbb{S}$, called the switching rule, is a right-continuous switching signal (cadlag). Let $\Xi = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying usual conditions. Let $\mathbf{w}(t) = [w_1(t),\dots,w_m(t)]^T$, $t \geqslant 0$ be an $\mathbb{R}^m$-valued vector standard Wiener process defined on the probability space Ξ. We assume that processes $w_k(t)$ and $\sigma(t)$ are mutually independent and both are $\{\mathcal{F}_t\}_{t\geqslant 0}$ adapted.

Let us consider the stochastic hybrid system described by the vector Itô differential equation

$$d\mathbf{x}(t) = \mathbf{f}\big(\mathbf{x}(t),\sigma(t)\big)\,dt + \sum_{k=1}^{m} \mathbf{g}_k\big(\mathbf{x}(t),\sigma(t)\big)\,dw_k(t), \quad \mathbf{x}(t_0) = \mathbf{x_0},\ \sigma(t_0) = \sigma_0, \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector, $\mathbf{x_0} \in \mathbb{R}^n$ and $\sigma_0 \in \mathbb{S}$ are initial values, $t \in \mathbb{T}$, $\mathbf{f}(\mathbf{x}(t),\sigma(t))$ and $\mathbf{g}_k(\mathbf{x}(t),\sigma(t))$ are defined by sets of $\{f(\mathbf{x}(t),l)\}$ and $\{\mathbf{g}_k(\mathbf{x}(t),l)\}$, respectively i.e. $\mathbf{f}(\mathbf{x}(t),\sigma(t)) = \mathbf{f}(\mathbf{x}(t),l)$, $\mathbf{g}_k(\mathbf{x}(t),\sigma(t)) = \mathbf{g}_k(\mathbf{x}(t),l)$ for $\sigma(t) = l$. Functions $\mathbf{f}: \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^n$ and $\mathbf{g}_k: \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^n$ are such that $\mathbf{f}(\mathbf{0},l) = \mathbf{g}_k(\mathbf{0},l) = \mathbf{0}$, $\forall l \in \mathbb{S}$, $k = 1,\dots,m$ and there exist nonnegative constants K_l, $l \in \mathbb{S}$, such that

$$\big|\mathbf{f}(\mathbf{x},l)\big|^2 + \sum_{k=1}^{m}\big|\mathbf{g}_k(\mathbf{x},l)\big|^2 \leqslant K_l\big(1+|\mathbf{x}|^2\big), \quad \forall \mathbf{x} \in \mathbb{R}^n,\ \forall l \in \mathbb{S}, \tag{2}$$

$$\big|\mathbf{f}(\mathbf{x},l) - \mathbf{f}(\mathbf{y},l)\big| + \sum_{k=1}^{m}\big|\mathbf{g}_k(\mathbf{x},l) - \mathbf{g}_k(\mathbf{y},l)\big| \leqslant K_l|\mathbf{x}-\mathbf{y}|, \quad \forall \mathbf{x},\mathbf{y} \in \mathbb{R}^n,\ \forall l \in \mathbb{S}. \tag{3}$$

For any twice differentiable function $\phi(\cdot,l)$ the l-th process has a generator $\mathcal{L}_l$ (the Itô operator for the l-th subsystem of system (1)) given in every structure by

$$\mathcal{L}_l\phi(\mathbf{x},l) = \sum_{\mu=1}^{n} f^{\mu}(\mathbf{x},l)\frac{\partial\phi(\mathbf{x},l)}{\partial x_\mu} + \frac{1}{2}\sum_{r,s=1}^{n}\sum_{k=1}^{m} g_k^r(\mathbf{x},l)g_k^s(\mathbf{x},l)\frac{\partial^2\phi(\mathbf{x},l)}{\partial x_r\partial x_s}, \quad l \in \mathbb{S}. \tag{4}$$

We also consider a particular case of Eq. (1), where the switching rule $\sigma(t)$ is given by a right-continuous Markov chain $r(t)$ defined on the probability space Ξ and taking values in a finite state space $\mathbb{S} = \{1,2,\dots,N\}$ with a generator $\boldsymbol{\Gamma} = [\gamma_{ij}]_{N\times N}$, i.e.

$$\mathbb{P}\big\{r(t+\delta) = j \,\big|\, r(t) = i\big\} = \begin{cases} \gamma_{ij}\delta + o(\delta), & \text{if } i \neq j, \\ 1 + \gamma_{ii}\delta + o(\delta), & \text{if } i = j, \end{cases} \tag{5}$$

where $\delta > 0$, $\gamma_{ij} \geqslant 0$ is the transition rate from i to j if $i \neq j$, $\gamma_{ii} = -\sum_{i \neq j} \gamma_{ij}$. We assume that the Markov chain is irreducible i.e. $\operatorname{rank}(\boldsymbol{\Gamma}) = N - 1$, and has a unique stationary distribution $\boldsymbol{\mathcal{P}} = [\pi_1, \pi_2, \dots, \pi_N]^T \in \mathbb{R}^N$ which can be determined by solving

$$\begin{cases} \boldsymbol{\mathcal{P}\Gamma} = \mathbf{0}, \\ \text{subject to } \sum_{i=1}^N \pi_i = 1 \quad \text{and} \quad \pi_i > 0 \quad \text{for all } i \in \mathbb{S}. \end{cases} \tag{6}$$

The process $(\mathbf{x}(t), \sigma(t))$ for $\sigma(t) = r(t)$ has a generator $\mathcal{L}$ given as follows. For each $l \in \mathbb{S}$ and any twice differentiable function $\phi(\cdot, l)$,

$$\mathcal{L}\phi(\mathbf{x}, l) = \sum_{\mu=1}^{n} f^{\mu}(\mathbf{x}, l) \frac{\partial \phi(\mathbf{x}, l)}{\partial x_{\mu}} + \frac{1}{2} \sum_{r,s=1}^{n} \sum_{k=1}^{m} g_k^r(\mathbf{x}, l) g_k^s(\mathbf{x}, l) \frac{\partial^2 \phi(\mathbf{x}, l)}{\partial x_r \partial x_s} + \sum_{j=1}^{N} \gamma_{lj} \phi(\mathbf{x}, j), \quad l \in \mathbb{S}. \tag{7}$$

Definition 1. The null solution $\mathbf{x}(t) \equiv \mathbf{0}$ of the stochastic differential equation (1) is said to be p-th mean exponentially stable if there exists a pair of positive scalars α, c such that $\forall (\mathbf{x_0}, t_0) \in \mathbb{R}^n \times \mathbb{R}_+$

$$\mathbb{E}\big[\big|\mathbf{x}(t, \mathbf{x_0}, t_0)\big|^p\big] \leqslant c\mathbb{E}\big[|\mathbf{x_0}|^p\big] \exp\big\{-\alpha(t - t_0)\big\}, \quad t \geqslant t_0 \tag{8}$$

or if

$$\limsup_{t \to \infty} \frac{1}{t} \log\big(\mathbb{E}\big[\big|\mathbf{x}(t, \mathbf{x_0}, t_0)\big|^p\big]\big) \leqslant -\alpha, \quad t \geqslant t_0. \tag{9}$$

The left-hand side of (9) is called the p-th mean Lyapunov exponent of the solution of Eq. (1). In the case of $p = 2$ the null solution $\mathbf{x}(t) \equiv \mathbf{0}$ of the stochastic differential equation (1) is called exponential mean-square stable.

Definition 2. The null solution $\mathbf{x}(t) \equiv \mathbf{0}$ of the stochastic differential equation (1)

1. is stable in probability if, for any $t_0 \geqslant 0$ and $\epsilon > 0$

$$\lim_{\mathbf{x_0} \to 0} P\Big(\sup_{t_0 \leqslant t} \big|\mathbf{x}(t, \mathbf{x_0}, t_0)\big| > \epsilon\Big) = 0, \tag{10}$$

2. is asymptotically stable in probability if it is stable in probability and, for any $t_0 \geqslant 0$ and $\mathbf{x_0} \in \mathbb{R}^n$

$$P\Big(\lim_{t \to \infty} \big|\mathbf{x}(t, \mathbf{x_0}, t_0)\big| = 0\Big) = 1. \tag{11}$$

Now, we introduce theorems concerning the asymptotic stability in probability of the null solution $\mathbf{x}(t) \equiv \mathbf{0}$ of hybrid systems with any switching rule and with a Markovian switching rule.

It is well known that the stability of all subsystems of the hybrid system is not sufficient for the stability of the whole hybrid system [17]. Hence it follows that to establish the sufficient conditions of the stability of the null solution $\mathbf{x}(t) \equiv \mathbf{0}$ of hybrid systems with any switching we had to find an additional condition. To do it we use the common Lyapunov function approach.

Definition 3. A positive definite radially unbounded (i.e. $V : R^n \to R_+ \backslash \{0\}$ for $x \neq 0$ satisfying the condition $\lim_{R \to \infty} \inf_{|\mathbf{x}| < R} V(\mathbf{x}) = \infty$) twice differentiable function, $V(\mathbf{x}, l) = V(\mathbf{x})$ for all $l \in \mathbb{S}$, is called a common Lyapunov function for system (1) if

$$\mathcal{L}_l V(\mathbf{x}) < 0, \quad \forall \mathbf{x} \neq \mathbf{0}, \ \forall l \in \mathbb{S}, \tag{12}$$

where $\mathcal{L}_l(\cdot)$ is given by (4).

Theorem 1. *If there is a common Lyapunov function $V(\mathbf{x})$ for system* (1)*, then the null solution $\mathbf{x} \equiv \mathbf{0}$ of the hybrid system* (1) *is asymptotically stable in probability for any switching.*

The proof follows from Definition 3 and stochastic stability theory [16].

Sufficient conditions for the asymptotic stability in probability of the null solution $\mathbf{x} \equiv \mathbf{0}$ of the hybrid system (1) with a Markovian switching rule in terms of the Lyapunov function are given by the following theorem.

Theorem 2. *(See [15].) Let $D \subset \mathbb{R}^n$ be an open neighborhood of $\mathbf{0}$. Suppose that for each $l \in \mathbb{S}$, there exists a nonnegative function $V(\cdot, l) : D \to \mathbb{R}$ such that*

1. *$V(\cdot, l)$ is continuous in D and vanishes only at $\mathbf{x} = \mathbf{0}$;*
2. *$V(\cdot, l)$ is twice continuously differentiable in $D \backslash \{\mathbf{0}\}$ and satisfies*

$$\mathcal{L}V(\mathbf{x}, l) \leqslant 0 \ (< 0), \quad \forall \mathbf{x} \in D \backslash \{\mathbf{0}\}, \ \forall l \in \mathbb{S}. \tag{13}$$

Then the null solution $\mathbf{x} \equiv \mathbf{0}$ of the hybrid system (1) *is* (*asymptotically*) *stable in probability.*

3. Stabilizability of nonlinear hybrid systems

Let us consider the stochastic control hybrid system described by the vector Itô differential equations

$$d\mathbf{x}(t) = \mathbf{f}_0\big(\mathbf{x}(t), \sigma(t)\big)\,dt + \sum_{i=1}^{p} u_i\big(\sigma(t)\big)\mathbf{f}_i\big(\mathbf{x}(t), \sigma(t)\big) + \sum_{k=1}^{m} \mathbf{g}_k\big(\mathbf{x}(t), \sigma(t)\big)\,dw_k(t), \quad \mathbf{x}(t_0) = \mathbf{x_0}, \ \sigma(t_0) = \sigma_0, \tag{14}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector, $\mathbf{u} = [u_1, \ldots, u_p]^T$ is a measurable $\mathbb{R}^p$ – a real-valued control law, $\mathbf{x}_0 \in \mathbb{R}^n$ and $\sigma_0 \in \mathbb{S}$ are initial values, $t \in \mathbb{T}$. Functions $\mathbf{f}_i : \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^n$ and $\mathbf{g}_k : \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^n$ are smooth Lipschitz, $\mathbf{f}_i(\mathbf{0}, l) = \mathbf{g}_k(\mathbf{0}, l) = \mathbf{0}$, $\forall l \in \mathbb{S}$, $i = 0, \ldots, p$, $k = 1, \ldots, m$. For $\sigma(t) = l \in \mathbb{S}$ we have $\mathbf{f}_i(\mathbf{x}(t), \sigma(t)) = \mathbf{f}_i(\mathbf{x}(t), l)$, $\mathbf{g}_k(\mathbf{x}(t), \sigma(t)) = \mathbf{g}_k(\mathbf{x}(t), l)$, $u_i(\sigma(t)) = u_i(l)$. There exist nonnegative constants K_l, $l \in \mathbb{S}$, such that

$$\sum_{i=0}^{p} \big|\mathbf{f}_i(\mathbf{x}, l)\big|^2 + \sum_{k=1}^{m} \big|\mathbf{g}_k(\mathbf{x}, l)\big|^2 \leqslant K_l\big(1 + |\mathbf{x}|^2\big), \quad \forall \mathbf{x} \in \mathbb{R}^n, \ \forall l \in \mathbb{S}. \tag{15}$$

The aim of the paper is to establish sufficient conditions under which one can design a state feedback control law so that the null solution $\mathbf{x} \equiv \mathbf{0}$ of the hybrid control system (14) is asymptotically stable in probability. We extend results of Florchinger for stochastic nonhybrid systems to hybrid case [14,13].

Let us consider now a special case of the hybrid control system (14) with the switching signal given by a Markovian switching rule i.e. $\sigma(t) = r(t)$. We introduce the following notation of operators L and L_i,

$$L\phi(\mathbf{x}, l) = \sum_{\mu=1}^{n} f_0^{\mu}(\mathbf{x}, l)\frac{\partial \phi(\mathbf{x}, l)}{\partial x_\mu} + \frac{1}{2}\sum_{r,s=1}^{n}\sum_{k=1}^{m} g_k^r(\mathbf{x}, l) g_k^s(\mathbf{x}, l)\frac{\partial^2 \phi(\mathbf{x}, l)}{\partial x_r \partial x_s} + \sum_{j=1}^{N} \gamma_{lj}\phi(\mathbf{x}, j), \quad l \in \mathbb{S}, \tag{16}$$

$$L_i\phi(\mathbf{x}, l) = \sum_{\mu=1}^{n} f_i^{\mu}(\mathbf{x}, l)\frac{\partial \phi(\mathbf{x}, l)}{\partial x_\mu}, \quad i = 1, \ldots, p, \ l \in \mathbb{S}. \tag{17}$$

Then, the following stabilization result for the control hybrid system (14) holds.

Theorem 3. *Let $D \subset \mathbb{R}^n$ be an open neighborhood of $\mathbf{0}$. Suppose that for each $l \in \mathbb{S}$, there exists a nonnegative function $V(\cdot, l) : D \to \mathbb{R}$ such that*

1. *$V(\cdot, l)$ is continuous in D and vanishes only at $\mathbf{x} = \mathbf{0}$;*
2. *$V(\cdot, l)$ is twice continuously differentiable in $D \backslash \{\mathbf{0}\}$ and satisfies*

$$LV(\mathbf{x}, l) < 0, \quad \forall \mathbf{x} \in D \backslash \{\mathbf{0}\}, \ \forall l \in \mathbb{S}. \tag{18}$$

Then the control law $\mathbf{u} : \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^p$ given by

$$u_i\big(\mathbf{x}(t), l\big) = -L_i V(\mathbf{x}, l), \quad i = 1, \ldots, p, \ l \in \mathbb{S} \tag{19}$$

renders the null solution $\mathbf{x} \equiv \mathbf{0}$ of the stochastic hybrid system (14) *asymptotically stable in probability for a Markovian switching rule $r(t)$.*

Proof. Applying the infinitesimal operator $\mathcal{L}$ defined by (7) to the hybrid system (14) we find

$$\mathcal{L}V(\mathbf{x}, l) = LV(\mathbf{x}, l) - \sum_{i=1}^{p}\big(L_i V(\mathbf{x}, l)\big)^2, \quad \forall \mathbf{x} \in D \backslash \{\mathbf{0}\}, \ \forall l \in \mathbb{S}. \tag{20}$$

Taking into account hypothesis (18) we obtain

$$\mathcal{L}V(\mathbf{x}, l) < 0, \quad \forall \mathbf{x} \in D \backslash \{\mathbf{0}\}, \ \forall l \in \mathbb{S} \tag{21}$$

and according to Theorem 2 the null solution $\mathbf{x}(t) \equiv \mathbf{0}$ of hybrid system (14) with a Markovian switching rule $r(t)$ is asymptotically stable in probability. □

Remark 1. In the case of the existence of a common function, i.e. $V(\mathbf{x},l)=V(\mathbf{x})$ for every $l\in\mathbb{S}$, we can formulate the sufficient conditions for the asymptotic stability in probability of the null solution $\mathbf{x}\equiv\mathbf{0}$ of the hybrid systems (14) with any switching using Theorem 1. Operators (16) and (17) in this case are given by

$$L_l\phi(\mathbf{x})=\sum_{\mu=1}^{n}f_0^{\mu}(\mathbf{x},l)\frac{\partial\phi(\mathbf{x})}{\partial x_\mu}+\frac{1}{2}\sum_{r,s=1}^{n}\sum_{k=1}^{m}g_k^r(\mathbf{x},l)g_k^s(\mathbf{x},l)\frac{\partial^2\phi(\mathbf{x})}{\partial x_r\partial x_s},\quad l\in\mathbb{S}, \tag{22}$$

$$(L_i)_l\phi(\mathbf{x})=\sum_{\mu=1}^{n}f_i^{\mu}(\mathbf{x},l)\frac{\partial\phi(\mathbf{x})}{\partial x_\mu},\quad i=1,\dots,p,\ l\in\mathbb{S}. \tag{23}$$

From Theorem 1 it follows that if the common positive define, radially unbounded and twice differentiable function $V(\mathbf{x})$ exists for the hybrid system (14) such that

$$L_lV(\mathbf{x})<0,\quad \forall\mathbf{x}\neq\mathbf{0},\ \forall l\in\mathbb{S}, \tag{24}$$

then the control law $\mathbf{u}:\mathbb{R}^n\times\mathbb{S}\to\mathbb{R}^p$ given by

$$u_i\big(\mathbf{x}(t),l\big)=-(L_i)_lV(\mathbf{x}),\quad i=1,\dots,p,\ l\in\mathbb{S} \tag{25}$$

renders the null solution $\mathbf{x}\equiv\mathbf{0}$ of the hybrid system (14) asymptotically stable in probability for any switching.

Example 1. Let us consider the nonlinear hybrid system of a form (14) with $\sigma(t)=r(t)$, $n=2$, $N=3$, $m=1$, $p=1$ given as follows

$$d\mathbf{x}(t)=\begin{bmatrix}-5x_1-\frac{1}{2}x_1\sin x_1\\ -3x_2\end{bmatrix}dt+u_1(1)\begin{bmatrix}x_2\\ 0\end{bmatrix}dt+\begin{bmatrix}x_2\\ x_1\end{bmatrix}dw(t),\quad \text{for } l=1, \tag{26}$$

$$d\mathbf{x}(t)=\begin{bmatrix}\frac{1}{2}x_1\\ \frac{1}{4}x_2\sin x_2\end{bmatrix}dt+u_1(2)\begin{bmatrix}x_1\\ x_2\end{bmatrix}dt+\begin{bmatrix}x_2+\frac{1}{3}x_2\sin x_2\\ 0\end{bmatrix}dw(t),\quad \text{for } l=2, \tag{27}$$

$$d\mathbf{x}(t)=\begin{bmatrix}x_1\\ x_2\sin x_1\end{bmatrix}dt+u_1(3)\begin{bmatrix}0\\ x_1\end{bmatrix}dt+\begin{bmatrix}0\\ x_1-\frac{1}{2}x_1\sin^2 x_1\end{bmatrix}dw(t),\quad \text{for } l=3 \tag{28}$$

with the matrix $\boldsymbol{\Gamma}$ given by

$$\boldsymbol{\Gamma}=\begin{bmatrix}-3&1&2\\ 2&-2&0\\ 4&0&-4\end{bmatrix}. \tag{29}$$

Let us take functions $V(\mathbf{x},1)=\frac{1}{2}x_1^2+\frac{1}{2}x_2^2$, $V(\mathbf{x},2)=x_1^2+2x_2^2$, $V(\mathbf{x},3)=2x_1^2+x_2^2$. Then

$$LV(\mathbf{x},1)=-x_1^2\left(1+\frac{1}{2}\sin x_1\right)<0, \tag{30}$$

$$LV(\mathbf{x},2)=x_2^2\left(\sin x_2+\left(1+\frac{1}{3}\sin x_2\right)^2-3\right)<0, \tag{31}$$

$$LV(\mathbf{x},3)=x_1^2\left(\left(1-\frac{1}{2}\sin^2 x_1\right)^2-2\right)+2x_2^2(\sin x_1-1)<0 \tag{32}$$

for all $\mathbf{x}\neq 0$. According to Theorem 3 we obtain that the controller $\mathbf{u}(\mathbf{x}(t),r(t))$ given by

$$\mathbf{u}\big(\mathbf{x},r(t)\big)=\begin{cases}-x_1x_2 & \text{for } r(t)=1,\\ -2x_1^2-4x_2^2 & \text{for } r(t)=2,\\ -2x_1x_2 & \text{for } r(t)=3\end{cases} \tag{33}$$

renders the null solution $\mathbf{x}\equiv\mathbf{0}$ of the hybrid system (26)–(29) asymptotically stable in probability.

An exemplary simulation is shown in Fig. 1.

4. Stabilizability of bilinear hybrid systems

Let us consider now the bilinear hybrid system described by the vector Itô stochastic differential equation

$$d\mathbf{x}(t)=\left[\mathbf{A}\big(\sigma(t)\big)\mathbf{x}(t)+\sum_{i=1}^{p}\mathbf{u}_i\big(\sigma(t)\big)\mathbf{C}_i\big(\sigma(t)\big)\mathbf{x}(t)\right]dt+\sum_{k=1}^{m}\mathbf{B}_k\big(\sigma(t)\big)\mathbf{x}(t)\,dw_k(t),\quad \mathbf{x}(t_0)=\mathbf{x_0},\ \sigma(t_0)=\sigma_0, \tag{34}$$

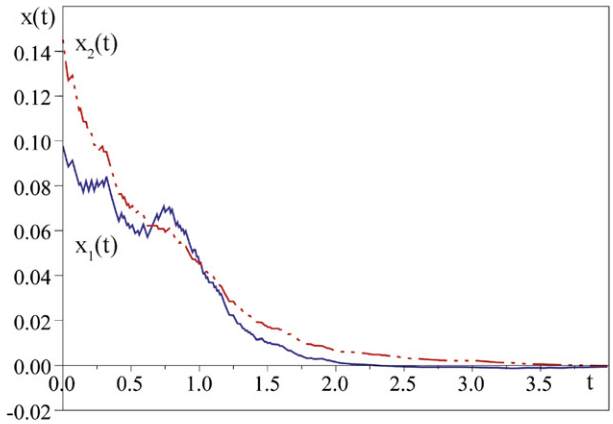

Fig. 1. An example of asymptotically stable samples $x_1(t)$ and $x_2(t)$ for system (26)–(29).

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector, $\mathbf{u} = [u_1, \ldots, u_p]^T \in \mathbb{R}^p$, $t \in \mathbb{T}$, $\mathbf{x_0} \in \mathbb{R}^n$ and $\sigma_0 \in \mathbb{S}$ are initial values, matrices $\mathbf{A}(\cdot), \mathbf{C}_i(\cdot), \mathbf{B}_k(\cdot) : \mathbb{S} \to \mathbb{R}^{n\times n}$, $i = 1, \ldots, p$, $k = 1, \ldots, m$. For $\sigma(t) = l$ we have $\mathbf{A}(\sigma(t)) = \mathbf{A}(l), \mathbf{C}_i(\sigma(t)) = \mathbf{C}_i(l), \mathbf{B}_k(\sigma(t)) = \mathbf{B}_k(l)$, $u_i(\sigma(t)) = u_i(l)$.

4.1. Open-loop controls

At the beginning of this section we remind some facts concerning the stability of linear hybrid systems with any switching $\sigma(t)$ (a special case of system (34)) i.e.,

$$d\mathbf{x}(t) = \mathbf{A}\big(\sigma(t)\big)\mathbf{x}(t)\,dt + \sum_{k=1}^{m} \mathbf{B}_k\big(\sigma(t)\big)\mathbf{x}(t)\,dw_k(t), \quad \mathbf{x}(t_0) = \mathbf{x_0},\ \sigma(t_0) = \sigma_0. \tag{35}$$

In this particular case we can formulate a definition of a common quadratic Lyapunov function and the theorem similar to Definition 3 and Theorem 1, respectively.

Definition 4. If there is a common definite matrix $\mathbf{P}$ satisfying

$$\mathbf{A}(l)^T\mathbf{P} + \mathbf{P}\mathbf{A}(l) + \sum_{k=1}^{m} \mathbf{B}_k(l)^T\mathbf{P}\mathbf{B}_k(l) < 0, \quad \forall l \in \mathbb{S}, \tag{36}$$

then $V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x}$ is called the common quadratic Lyapunov function for all subsystems of system (35).

Theorem 4. *(See [24].) If there is the common quadratic Lyapunov function for all subsystems of* (35)*, then the null solution* $\mathbf{x} \equiv \mathbf{0}$ *of the hybrid system* (35) *is exponentially mean-square stable for any switching.*

The application of the common quadratic Lyapunov function in Theorem 4 establishes sufficient conditions for the exponential mean-square stability for the null solution $\mathbf{x} \equiv \mathbf{0}$ of the linear hybrid system with any switching.

We remind now some basic facts concerning the Lie algebra [6].

Definition 5. A Lie algebra $\mathbb{L}$ over a field $\mathbb{R}$ is a triple $(V, +, [\cdot,\cdot])$, where $(V, +)$ is a vector space over a field $\mathbb{R}$ and $[\cdot,\cdot] : V \times V \to V$ is a bilinear mapping such that

1. $[\mathbf{v}_1, \mathbf{v}_2] = -[\mathbf{v}_2, \mathbf{v}_1]$,
2. $[\mathbf{v}_1, [\mathbf{v}_2, \mathbf{v}_3]] + [\mathbf{v}_2, [\mathbf{v}_3, \mathbf{v}_1]] + [\mathbf{v}_3, [\mathbf{v}_1, \mathbf{v}_2]] = \mathbf{0}$.

For the Lie algebra of matrices we have $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$, $\forall \mathbf{A}, \mathbf{B} \in \mathbb{L}$. $\mathbb{L}(\mathbf{A}_1, \ldots, \mathbf{A}_r)$ denotes a Lie algebra generated by matrices $\mathbf{A}_1, \ldots, \mathbf{A}_r$.

Definition 6. For every Lie algebra $\mathbb{L}$ we find

$$\begin{aligned} &\mathbb{L}^0 = \mathbb{L}, \\ &\quad\vdots \\ &\mathbb{L}^{(n+1)} = \big[\mathbb{L}^n, \mathbb{L}^n\big]. \end{aligned} \tag{37}$$

We say that the Lie algebra $\mathbb{L}$ is solvable if $\mathbb{L}^n = \{\mathbf{0}\}$ for some n.

Lemma 1. *(See [22].) A Lie algebra of matrices $\mathbb{L}(\mathbf{A}_1, \ldots, \mathbf{A}_r)$ is solvable iff there exists a nonsingular matrix* $\mathbf{M}$ *such that matrices of a form* $\mathbf{M}\mathbf{A}_i\mathbf{M}^{-1}$ *are upper-triangular for every matrix* $\mathbf{A}_i \in \mathbb{L}$, $1 \leqslant i \leqslant r$.

Another sufficient conditions of the exponential mean-square stability for hybrid systems can be proposed for linear hybrid systems with any switching with a special structure defined by the Lie algebra generated by the matrices $\mathbf{A}(l), \mathbf{B}_k(l)$, $k = 1, \ldots, m$, $l \in \mathbb{S}$.

Theorem 5. *(See [24].) If the Lie algebra* $\mathbb{L}(\mathbf{A}(l), \mathbf{B}_k(l),\ l \in \mathbb{S},\ k = 1, \ldots, m)$ *is solvable and furthermore*

$$2\,\mathrm{Re}\big(\lambda_j\big(\mathbf{A}(l)\big)\big) + \sum_{k=1}^{m} \big|\lambda_j\big(\mathbf{B}_k(l)\big)\big|^2 < 0, \quad j = 1, \ldots, n,\ \forall l \in \mathbb{S}, \tag{38}$$

then the hybrid system (35) *is exponentially mean-square stable for any switching.*

Remark 2. If all eigenvalues of matrices $\mathbf{B}_k(l)$, $k = 1, \ldots, m$, are real, then condition (38) is equivalent to the Hurwitz character of all matrices $2\mathbf{A}(l) + \sum_{k=1}^{m} \mathbf{B}_k^2(l)$, $l \in \mathbb{S}$.

We note that Theorem 5 is very useful in many switching control problems. We consider a particular case of Eq. (34) for $p = 1$ i.e.,

$$d\mathbf{x}(t) = \big[\mathbf{A}\big(\sigma(t)\big)\mathbf{x}(t) + \mathbf{u}\big(\sigma(t)\big)\mathbf{C}\big(\sigma(t)\big)\mathbf{x}(t)\big]\,dt + \sum_{k=1}^{m} \mathbf{B}_k\big(\sigma(t)\big)\mathbf{x}(t)\,dw_k(t), \quad \mathbf{x}(t_0) = \mathbf{x_0},\ \sigma(t_0) = \sigma_0 \tag{39}$$

and we assume that the Lie algebra

$$\mathbb{L}\big(\mathbf{A}(l), \mathbf{B}_k(l), \mathbf{C}(l),\ l \in \mathbb{S},\ k = 1, \ldots, m\big) \tag{40}$$

is solvable.

To stabilize system (39) we seek a piecewise constant hybrid controller i.e.

$$\mathbf{u}\big(\sigma(t)\big) = \beta\big(\sigma(t)\big), \quad \beta(\cdot) : \mathbb{S} \to \mathbb{R}, \tag{41}$$

such that the null solution $\mathbf{x} \equiv \mathbf{0}$ of the hybrid system

$$d\mathbf{x}(t) = \big[\mathbf{A}\big(\sigma(t)\big) + \beta\big(\sigma(t)\big)\mathbf{C}\big(\sigma(t)\big)\big]\mathbf{x}(t)\,dt + \sum_{k=1}^{m} \mathbf{B}_k\big(\sigma(t)\big)\mathbf{x}(t)\,dw_k(t), \quad \mathbf{x}(t_0) = \mathbf{x_0},\ \sigma(t_0) = \sigma_0 \tag{42}$$

is exponentially mean-square stable. Note that a similar model for nonhybrid systems has been analyzed in [18].

Since the Lie algebra $\mathbb{L}$ given by (40) is solvable, then there exists a matrix $\mathbf{M}$ (Lemma 1) that simultaneously triangularizes matrices $\mathbf{A}(l)$, $\mathbf{C}(l)$, $\mathbf{B}_k(l)$, i.e., $\tilde{\mathbf{A}}(l) = \mathbf{M}\mathbf{A}(l)\mathbf{M}^{-1}$, $\tilde{\mathbf{B}}_k(l) = \mathbf{M}\mathbf{B}_k(l)\mathbf{M}^{-1}$, $k = 1, \ldots, m$, $\tilde{\mathbf{C}}(l) = \mathbf{M}\mathbf{C}(l)\mathbf{M}^{-1}$ are upper-triangular for every $l \in \mathbb{S}$. In this case the set of eigenvalues of system matrices is given by

$$\lambda\big(\mathbf{A}(l) + \beta(l)\mathbf{C}(l)\big) = \big\{\tilde{a}_{jj}(l) + \beta(l)\tilde{c}_{jj}(l),\ j = 1, \ldots, n\big\}, \quad l \in \mathbb{S}, \tag{43}$$

$$\lambda\big(\mathbf{B}_k(l)\big) = \big\{\tilde{b}_{jj}^{k}(l),\ j = 1, \ldots, n\big\}, \quad k = 1, \ldots, m,\ l \in \mathbb{S}. \tag{44}$$

Using Theorem 5 in order to stabilize system (42) we can find a set of stabilizing controllers given as follows

$$\Gamma_l = \left\{\beta(l) \in \mathbb{R}\colon\ \mathrm{Re}\big(\tilde{a}_{jj}(l) + \beta(l)\tilde{c}_{jj}(l)\big) + \frac{1}{2}\sum_{k=1}^{m} \big|\tilde{b}_{jj}^{k}(l)\big|^2 < 0,\ j = 1, \ldots, n\right\}, \quad l \in \mathbb{S}. \tag{45}$$

They are given by n linear restrictions for every $l \in \mathbb{S}$.

Example 2. Let us consider the hybrid system (39) for $n = 2$, $N = 2$, $m = 1$, $p = 1$ defined by the following matrices

$$\mathbf{A}(1) = \begin{bmatrix} -0.5 & 0.1 \\ 0.1 & -0.5 \end{bmatrix}, \qquad \mathbf{C}(1) = \begin{bmatrix} -0.4 & 0.3 \\ 0.3 & -0.4 \end{bmatrix}, \qquad \mathbf{B}_1(1) = \begin{bmatrix} -1 & 1 \\ 3 & 1 \end{bmatrix}, \tag{46}$$

$$\mathbf{A}(2) = \begin{bmatrix} 0.5 & 0.1 \\ 0.1 & 0.5 \end{bmatrix}, \qquad \mathbf{C}(2) = \begin{bmatrix} -0.2 & 0.1 \\ 0.1 & -0.2 \end{bmatrix}, \qquad \mathbf{B}_1(2) = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}. \tag{47}$$

The Lie algebra $\mathbb{L}(\mathbf{A}(l), \mathbf{B}_1(l), \mathbf{C}(l),\ l = 1, 2)$ is solvable with the matrix $\mathbf{M}$ given by

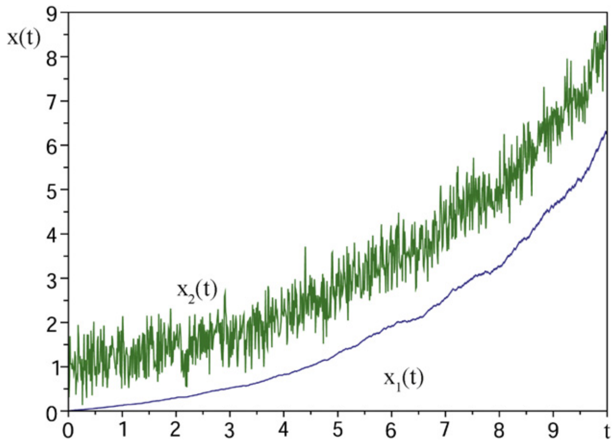

Fig. 2. Samples $x_1(t)$ and $x_2(t)$ of the hybrid system (34) with $\mathbf{u} \equiv 0$.

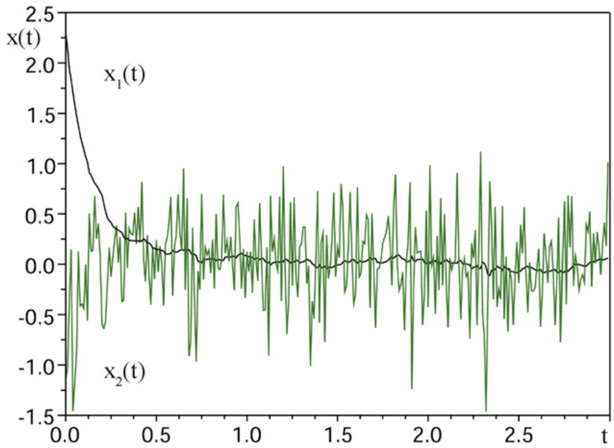

Fig. 3. Samples $x_1(t)$ and $x_2(t)$ of the hybrid system (34) with $\mathbf{u}$ given by (51).

$$\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \tag{48}$$

so we can stabilize system (46)–(47) by a piecewise constant hybrid controller (41). To determine a set of stabilizing controllers (45) we must solve the following algebraic linear inequalities

$$\begin{cases} -0.6 - 0.7\beta(1) + 2 < 0, \\ -0.4 - 0.1\beta(1) + 2 < 0, \end{cases} \quad \Rightarrow \quad \beta(1) > 16, \tag{49}$$

$$\begin{cases} 0.6 - 0.1\beta(2) + 2 < 0, \\ 0.4 - 0.3\beta(2) + 2 < 0, \end{cases} \quad \Rightarrow \quad \beta(2) > 26 \tag{50}$$

and a set of stabilizing controllers is given by

$$\Gamma_1 = \{\beta(1) \in \mathbb{R}\colon\ \beta(1) > 16\}, \tag{51}$$

$$\Gamma_2 = \{\beta(2) \in \mathbb{R}\colon\ \beta(2) > 26\}. \tag{52}$$

Exemplary simulations are shown in Figs. 2–3.

4.2. Closed-loop controls

Let us consider a special case of the hybrid system (14) given by bilinear structures described by (34).
Let us denote

$$\mu_l = \lambda_{\max}\left(\frac{\mathbf{A}(l) + \mathbf{A}(l)^T}{2}\right) + \frac{1}{2}\sum_{k=1}^{m} \lambda_{\max}\left(\mathbf{B}_k(l)\mathbf{B}_k^T(l)\right), \quad l \in \mathbb{S}. \tag{53}$$

Operators (16) and (17) reduce to the following ones

$$L\phi(\mathbf{x},l) = \mathbf{x}^T\mathbf{A}^T(l)\nabla\phi(\mathbf{x},l) + \frac{1}{2}\operatorname{tr}\big(\mathbf{B}_k(l)\mathbf{x}\mathbf{x}^T\mathbf{B}_k(l)^T\nabla^2\phi(\mathbf{x},l)\big) + \sum_{j=1}^{N}\gamma_{lj}\phi(\mathbf{x},j), \quad l \in \mathbb{S}, \tag{54}$$

$$L_i\phi(\mathbf{x},l) = \mathbf{x}^T\mathbf{C}_i^T(l)\nabla\phi(\mathbf{x},l), \quad i = 1,\dots,p,\ l \in \mathbb{S}, \tag{55}$$

where $\nabla\phi(\cdot,l)$ and $\nabla^2\phi(\cdot,l)$ denote the gradient and the Hessian of $\phi(\cdot,l)$, respectively.

Hence, using results of Khasminskii, Zhu and Yin [15] and Theorem 3 we can formulate for this special case the following theorem.

Theorem 6. *If there exists a positive solution $\mathbf{c} = [c_1, c_2, \dots, c_N]^T$, $c_l > 0$, for the linear system of algebraic equations given by*

$$\mu_l c_l + \frac{1}{2}\sum_{l=1}^{N}\gamma_{lj}c_j = \beta, \quad \forall l \in \mathbb{S}, \tag{56}$$

where $\beta = \sum_{l=1}^{N}\pi_l\mu_l c_l$ and the following inequality

$$\beta < 0 \tag{57}$$

is satisfied, then the control law $\mathbf{u} = [u_1, \dots, u_p]^T$ defined by

$$u_i(\mathbf{x},l) = -c_l\mathbf{x}^T\mathbf{C}_i(l)^T\mathbf{x}, \quad i = 1,\dots,p,\ l \in \mathbb{S} \tag{58}$$

renders the null solution $\mathbf{x} \equiv \mathbf{0}$ of the bilinear hybrid system (34) *asymptotically stable in probability for a Markovian switching rule.*

Proof. Note that for any matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{B}_k(l) \in \mathbb{R}^{n\times n}$, $k = 1,\dots,m$, $l \in \mathbb{S}$ we have

$$\operatorname{tr}\big(\mathbf{B}_k(l)\mathbf{x}\mathbf{x}^T\mathbf{B}_k(l)^T\mathbf{A}\big) = \mathbf{x}^T\mathbf{B}_k(l)^T\mathbf{A}\mathbf{B}_k(l)\mathbf{x}. \tag{59}$$

Let us consider the linear system of algebraic equations given by (56). Since $\operatorname{rank}(\boldsymbol{\Gamma}) = N-1$ and

$$\mathcal{P}^T\begin{bmatrix}\mu_1c_1 - \beta\\ \mu_2c_2 - \beta\\ \dots\\ \mu_Nc_N - \beta\end{bmatrix} = \mathbf{0} \tag{60}$$

then system (56) has infinite number of solutions. If there exists a positive solution $\mathbf{c} = [c_1, c_2, \dots, c_N]^T$, $c_l > 0$, then for a function $V(\mathbf{x},l) = \frac{1}{2}c_l|\mathbf{x}|^2$ we obtain

$$\begin{aligned}
LV(\mathbf{x},l) &= c_l\mathbf{x}^T\mathbf{A}(l)^T\mathbf{x} + \frac{1}{2}\sum_{k=1}^{m}\operatorname{tr}\big(\mathbf{B}_k(l)\mathbf{x}\mathbf{x}^T\mathbf{B}_k(l)^Tc_l\mathbf{I}\big) + \frac{1}{2}\sum_{j=1}^{N}\gamma_{lj}c_j|\mathbf{x}|^2\\
&= c_l\mathbf{x}^T\mathbf{A}(l)^T\mathbf{x} + \frac{1}{2}c_l\sum_{k=1}^{m}\mathbf{x}^T\mathbf{B}_k(l)^T\mathbf{B}_k(l)\mathbf{x} + \frac{1}{2}\sum_{j=1}^{N}\gamma_{lj}c_j|\mathbf{x}|^2\\
&= |\mathbf{x}|^2\left(c_l\frac{\mathbf{x}^T(\mathbf{A}(l)+\mathbf{A}(l)^T)\mathbf{x}}{2|\mathbf{x}|^2} + \frac{1}{2}c_l\sum_{k=1}^{m}\frac{\mathbf{x}^T\mathbf{B}_k(l)^T\mathbf{B}_k(l)\mathbf{x}}{|\mathbf{x}|^2} + \frac{1}{2}\sum_{j=1}^{N}\gamma_{lj}c_j\right)\\
&\leqslant |\mathbf{x}|^2\left(c_l\mu_l + \frac{1}{2}\sum_{j=1}^{N}\gamma_{lj}c_j\right) = |\mathbf{x}|^2\beta < 0, \quad \text{for } \mathbf{x} \neq \mathbf{0},
\end{aligned} \tag{61}$$

where the inequalities follow from (56) and (57), respectively. From Theorem 3 it follows that the control law $\mathbf{u}: \mathbb{R}^p \times \mathbb{S} \to \mathbb{R}$ defined by

$$u_i(\mathbf{x},l) = -c_l\mathbf{x}^T\mathbf{C}_i(l)^T\mathbf{x}, \quad i = 1,\dots,p,\ l \in \mathbb{S} \tag{62}$$

renders the null solution $\mathbf{x} \equiv \mathbf{0}$ of the bilinear hybrid system (34) asymptotically stable in probability for a Markovian switching rule.

Note that the necessary condition for satisfying the inequality $\beta < 0$ is $2\mu_l + \gamma_{ll} < 0$ for all $l \in \mathbb{S}$. □

Remark 3. Similar to the nonlinear case (Remark 1) we can formulate sufficient conditions for the asymptotic stabilizability of the null solution $\mathbf{x} \equiv \mathbf{0}$ of the bilinear hybrid system (34) for any switching. In the case of the existence of a common function, $V(\mathbf{x}, l) = V(\mathbf{x}) = \frac{1}{2}|\mathbf{x}|^2$ for all $l \in \mathbb{S}$, we obtain the following result. If

$$\mu_l < 0, \quad \forall l \in \mathbb{S}, \tag{63}$$

then the control law $\mathbf{u} : \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}$ defined by

$$u_i(\mathbf{x}, l) = -\mathbf{x}^T \mathbf{C}_i(l)\mathbf{x}, \quad i = 1, \ldots, p, \ l \in \mathbb{S} \tag{64}$$

renders the null solution $\mathbf{x} \equiv \mathbf{0}$ of the bilinear hybrid system (34) asymptotically stable in probability for any switching and the stabilizability problem is independent of the switching rule.

Proof. We consider a common function $V(\mathbf{x}) = \frac{1}{2}|\mathbf{x}|^2$ for all $l \in \mathbb{S}$. Then

$$\begin{aligned}
L_l V(\mathbf{x}) &= \mathbf{x}^T \mathbf{A}(l)^T \mathbf{x} + \frac{1}{2}\sum_{k=1}^{m} \operatorname{tr}\big(\mathbf{B}_k(l)\mathbf{x}\mathbf{x}^T \mathbf{B}_k(l)^T I\big) \\
&= \mathbf{x}^T \left(\frac{\mathbf{A}(l) + \mathbf{A}(l)^T}{2}\right)\mathbf{x} + \frac{1}{2}\sum_{k=1}^{m} \mathbf{x}^T \mathbf{B}_k(l)^T \mathbf{B}_k(l)\mathbf{x} \\
&= |\mathbf{x}|^2 \left(\mathbf{x}^T \left(\frac{\mathbf{A}(l) + \mathbf{A}(l)^T}{2|\mathbf{x}|^2}\right)\mathbf{x} + \frac{1}{2}\sum_{k=1}^{m} \frac{\mathbf{x}^T \mathbf{B}_k(l)^T \mathbf{B}_k(l)\mathbf{x}}{|\mathbf{x}|^2}\right) \\
&\leqslant |\mathbf{x}|^2 \left(\lambda_{\max}\left(\frac{\mathbf{A}(l) + \mathbf{A}(l)^T}{2}\right) + \frac{1}{2}\sum_{k=1}^{m} \lambda_{\max}\big(\mathbf{B}_k(l)^T \mathbf{B}_k(l)\big)\right) \\
&= |\mathbf{x}|^2 \mu_l < 0 \quad \text{for } \mathbf{x} \neq \mathbf{0}, \ \forall l \in \mathbb{S}
\end{aligned} \tag{65}$$

which follows from (63). From Remark 1 we obtain that the control law $\mathbf{u} : \mathbb{R}^p \times \mathbb{S} \to \mathbb{R}$ given by

$$u_i(\mathbf{x}, l) = -(L_i)_l V(\mathbf{x}) = -\mathbf{x}^T \mathbf{C}_i(l)^T \mathbf{x}, \quad i = 1, \ldots, p, \ l \in \mathbb{S} \tag{66}$$

renders the null solution $\mathbf{x} \equiv \mathbf{0}$ of the bilinear hybrid system (34) asymptotically stable in probability for any switching. $\square$

Example 3. Let us consider the bilinear hybrid system described by (34) with $\sigma(t) = r(t)$, $n = 2$, $N = 3$, $m = 2$, $p = 2$, given as follows

$$\begin{aligned}
d\mathbf{x}(t) &= \left(\begin{bmatrix} -2 & -1 \\ -0.75 & -2 \end{bmatrix}\mathbf{x}(t) + u_1(1)\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\mathbf{x}(t) + u_2(1)\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}\mathbf{x}(t)\right)dt + \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \end{bmatrix} dw_1(t), \quad \text{for } l = 1, \\
d\mathbf{x}(t) &= \left(\begin{bmatrix} 0.05 & 2 \\ -2 & 0.375 \end{bmatrix}\mathbf{x}(t) + u_1(2)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\mathbf{x}(t)\right)dt + \begin{bmatrix} 0 & -0.25 \\ 0.5 & 0 \end{bmatrix} dw_1(t), \quad \text{for } l = 2, \\
d\mathbf{x}(t) &= \left(\begin{bmatrix} 0.5 & 0.25 \\ 0.5 & -0.5 \end{bmatrix}\mathbf{x}(t) + u_1(3)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\mathbf{x}(t)\right)dt + \begin{bmatrix} -0.3 & 0.4 \\ 0.08 & 0.06 \end{bmatrix} dw_1(t) + \begin{bmatrix} -0.7 & 0.1 \\ 0.1 & 0.7 \end{bmatrix} dw_2(t), \quad \text{for } l = 3
\end{aligned} \tag{67}$$

and the matrix $\boldsymbol{\Gamma}$ given by (29).

Note that $\mu_1 = -2$, $\mu_2 = 0.5$, $\mu_3 = 1$, $\mathcal{P} = [0.5, 0.25, 0.25]^T$. Let us take functions $V(\mathbf{x}, l) = \frac{1}{2}c_l|\mathbf{x}|^2$, $l = 1, 2, 3$, with $c_l > 0$ given by algebraic equations (56) in the form

$$\begin{cases} -5c_1 + 0.75c_2 + 1.5c_3 = 0, \\ 4c_1 - 1.25c_2 - 0.5c_3 = 0, \\ 6c_1 - 0.25c_2 - 2.5c_3 = 0. \end{cases} \tag{68}$$

Solving (68) we obtain $c_2 = \frac{7}{3}c_1$, $c_3 = \frac{13}{6}c_1$, $c_1 > 0$ and

$$\beta = -c_1 + 0.125c_2 + 0.25c_3 = -\frac{1}{6}c_1 < 0. \tag{69}$$

According to Theorem 6 we obtain that the control $\mathbf{u}(\mathbf{x}, r(t))$ given by

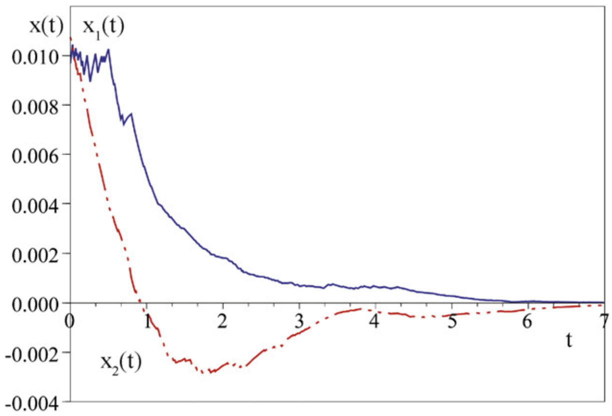

Fig. 4. An example of asymptotically stable in probability samples $x_1(t)$, $x_2(t)$ for system (67).

$$u_1\big(\mathbf{x}, r(t)\big) = \begin{cases} -c_1(x_1^2 + 4x_1x_2 + x_2^2) & \text{for } r(t) = 1, \\ -c_2x_1x_2 & \text{for } r(t) = 2, \\ -c_3x_1^2 & \text{for } r(t) = 3, \end{cases} \tag{70}$$

$$u_2\big(\mathbf{x}, r(t)\big) = \begin{cases} -c_1(2x_1^2 + 4x_1x_2 + x_2^2) & \text{for } r(t) = 1, \\ 0 & \text{for } r(t) = 2, \\ 0 & \text{for } r(t) = 3 \end{cases} \tag{71}$$

renders the null solution $\mathbf{x} \equiv \mathbf{0}$ of the hybrid system (67) asymptotically stable in probability.

An exemplary simulation is shown in Fig. 4.

5. Conclusions

In this paper nonlinear stochastic hybrid systems with two types of the switching rule: a Markovian or any switching rule have been analyzed. The results of [14] have been extended to stochastic hybrid systems and using Lyapunov technique [15] the sufficient conditions for the asymptotic stabilizability in probability have been derived. Also a stabilizing smooth state feedback law in every structure has been designed.

A special class of stochastic nonlinear hybrid systems, which are stochastic bilinear hybrid systems, have been considered more precisely. Two classes of bilinear systems: hybrid systems with any switching and hybrid systems with a Markovian switching rule have been discussed. The open- and closed-loop controller has been proposed for the exponential mean-square stabilizability and the asymptotic stabilizability in probability problem, respectively. Sufficient conditions for the stabilizability have been derived in both cases. In the case of hybrid systems with any switching rule sufficient conditions for the exponential mean-square stabilizability based on the Lie algebra theory [24] and also for the asymptotic stabilizability in probability based on the common Lyapunov function approach [17] have been formulated. While in the case of hybrid systems with a Markovian switching rule theorems which gives us sufficient conditions for the asymptotic stabilizability in probability using Lyapunov function methods [15] have been derived.

Finally, numerical examples have been given to illustrate the derived methods.

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